SHAPLEY COST ALLOCATION COINCIDES WITH RELATIVE STATUS: THE CASE OF SKYSCRAPERS

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Abstract

Empirical evidence shows that the value of units in a building generally rises with their floor level due to features such as the better view and lesser noise experienced in higher stories. We adopt a theoretical approach for examining the value of units in different floors based on the allocation of land and construction cost among the stories of the building. Relying on cooperative game theory analysis, we propose the Shapley value approach as a mechanism for allocating these costs. We examine the allocation mechanism and derive several closed-form properties by which the value pattern of stories in a building is rationalized. Furthermore, following Lakoff and Johnson (1980), we argue that agents achieve greater status from occupying higher stories because of inherent cognitive motives. We thus constitute the Relative L&J Status function and formally show that its properties coincide with those of the difference between the costs allocated to any two stories in the building, thereby, derive a new property to the Shapley solution. Finally, we empirically test the derived Shapley cost allocation properties and the attained results are consistent with our major predictions.

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1 Introduction

Conventional wisdom and existing empirical economic literature assert that the value of (otherwise identical) housing units in a building increases with their floor level. This phenomenon is generally explained by features such as the better view and lesser noise experienced in higher stories.[1] We propose a more fundamental aspect that rationalizes the price pattern of stories in buildings. This aspect, as will soon be clarified, stands at an intersection of cost allocation arguments that rely on both Shapley's solution to cooperative games [see Shapley (1953)] and Lakoff and Johnson's insight to orientational perceptions [see Lakoff and Johnson (1980) – hereafter, L&J].[2]

The framework we establish effectively proposes a normative—cost allocation based—ruling for the relative prices of stories in buildings. Particularly, it suggests a mechanism for allocating the land and construction cost among any number of stories that collectively constitute a given building.[3] Moreover, it offers a resolution to various puzzles in housing economics such as the willingness to pay different prices for identical units located on the same story of buildings that only differ in their number of stories.

The intuition for applying the Shapley approach for the allocation of the land and construction costs emerges from the observation according to which the construction of the higher stories in the building is utterly contingent upon the construction of the lower ones, while the construction of lower stories does not require the construction of the higher ones.[4] Hence, a cooperative game among the stories of the building arises.

A complementary rationale for the proposed cost allocation mechanism follows from L&J. By analyzing orientational metaphors, L&J argue and demonstrate that, normally, "up" (that is, higher stories in our case) associates with all that is perceived as

[1] See, for example, Benson *et al*. (1998), Wilhelmsson (2000) and Bourassa *et al*. (2004).

[2] L&J study is regarded as an imperative contribution to the topic of metaphors within the area of linguistics.

[3] Our model is thus also practical in resolving problems of real estate cost allocation within the firm, such as joint purchase of a building (or stories within a building) by a publicly traded firm and its controlling shareholders.

[4] This argument can also be stated from the standpoint of the entrepreneur, namely, that at the pre-construction stage, buyers of higher stories in the building impose (on the entrepreneur) harsher constraints than those imposed by buyers of lower stories. While by selling the first floor, for example, the entrepreneur assumes no obligation with respect to building additional stories on top of it, selling higher stories forces the entrepreneur to complete construction at least up to the highest floor sold. Thus, the higher the story that is sold prior to the completion of construction, the greater the inflexibility it entails on

positive, while "down" (lower stories in our case) connotes with everything, which is perceived as negative.

L&J demonstrate the manifestation of this phenomenon in the language. Several of the numerous examples are: happy is up and sad is down (as, for example, in "I'm feeling *up*," "He is *low* these days"); conscious is up and unconscious is down ("Wake *up*," "He *fell* asleep"); health and life are up and sickness and death are down ("Lazarus *rose* from the dead," "He *fell* ill"); having control or force is up and being subject to control or force is down ("I'm on *top* of the situation," "He's *under* my control"); more is up and less is down ("My income *rose* last year," "The number of errors he made is *low*"); high status is up and low status is down ("He has little *upward* mobility," "He's at the *bottom* of social hierarchy"); good is up and bad is down ("Things are looking *up*," "We hit a *peak* last year, but it's been *downhill* ever since"); virtue is up and depravity is down ("She is *high*-minded," "That would be *beneath* me"); and, finally, rational is up and emotional is down ("I *raised* the discussion back *up* to the rational plane," "He couldn't *rise above* his emotions").[5]

It follows from L&J that, *ceteris paribus*, the higher the floor one occupies, the more one is perceived (by oneself and others) to be happy, healthy, lively, good, rational, having control and force, having more, being in high status, etc. In other words, from a cognitive perspective (which may be either conscious or unconscious) occupying a higher story in a building is commonly preferred to occupying a lower one.[6] L&J thus rationalize the motivation for occupying higher stories, namely, that the relative status associated with a given story rises (falls) with the number of stories below (above) it.

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the entrepreneur due to the accompanying commitment to build all stories underneath, even if the realized demand is weak.

[5] In the context of our framework and following L&J, it is not surprising that, for example, the management level of many organizations frequently resides on the top floor of the building or, alternatively, if the organization does not have access to the top floor, then it is the highest available story that populates the management level (as commonly heard among peers: "I'm going *up* to the management level…"). While it is commonly believed that management acquires the highest possible floor due to the more attractive landscape, it follows from L&J that another important motivation corresponds to the cognitive connotations of the terms "up" and "down."

[6] Of course, the high (low) story receives its positive (negative) connotation from being higher (lower) than the stories underneath (above). This implies that, for example, occupying a top floor of a building with lower stories underneath is more valuable than occupying a floor with the same height, however, with no stories below.

In the general economic literature and specifically in the cooperative game theory literature, a number of methods have been proposed for allocating costs (utilities) among players within coalitions. Notably, Shapley (1953) presents a solution to the allocation problem that both conforms to conditions of fairness and is unique. Because of the inherent interdependence among stories in the case of vertical construction, we adopt the Shapley value approach by which we propose a mechanism for allocating the land and construction cost among the stories in the building.[7]

According to the Shapley solution, the cost distributed to each story rises with the story's level in the building in a particular way. In fact, as it turns out, each floor bears only its true pro rata share in the total cost associated with its construction.

Specifically, the cost of those elements in the building that commonly and equally serve all stories in the building (those element that embrace the fixed cost such as land, foundations of the structure, common infrastructure, etc.) and the marginal construction cost of the first floor are equally divided among all stories. However, the marginal construction cost of the second floor is equally allocated only among the floors beyond the first, and more generally, the marginal cost of the n-th floor is distributed only among the stories beyond the (n–1)-th floor.

Shapley (1953)[8] shows that the proposed solution conforms to the following properties: a) efficiency, i.e. the sum of the individual cost allocated to each player equals the total cost; b) symmetry, i.e. players that identically affect the cost function for every coalition bear the same cost; c) dummy player, i.e. players that do not affect the cost function for every coalition bear no cost; and d) additivity, i.e. for a given set of players, the Shapley value that attains for a sum of m games, m=1,2,…,M, is equal to the sum of the Shapley values separately solved for each game.

Note that while property (a) guarantees efficiency, such that all costs are allocated among all stories, properties (b) and (c) ensure that the solution conforms to standards of fairness. Property (d), which relates the Shapley values derived from different games, guarantees the uniqueness of the solution. Hence, the power of the Shapely approach is

[7] Moulin and Shenker (1992) also propose a cost sharing mechanism that somewhat resembles the Shapley solution. In their model, however, the agents share a one-input and one-output technology with decreasing returns.

in, among others, maintaining a unique solution that simultaneously obeys to the above set of axioms.

Following L&J, we also establish the *Relative L&J Status* function that sustains a set of reasonable status properties. We then show that the difference in the cost allocated to any two stories in a given building (resulting from the Shapley approach) further exhibits those same properties of the Relative L&J Status function, thereby, establishing a new property for the Shapley solution.[9]

Finally, we conduct an empirical test of the major predictions attained under the Shapley cost allocation. The empirical results support the cost allocation properties according to which the price ratio between any two stories j and i ($j>i$) increases with j and decreases with i, and the ratio drops with the number of stories in the building

The interest in the Shapley solution to allocation problems emerges for various reasons. First, it has been shown that under suitable conditions (assumptions regarding the utility function and the endowments and given a continuum of agents), every Shapley value allocation coincides with a competitive solution [see Shapley (1964), Aumann and Shapely (1974), Aumann (1975), and Mas-Colell (1977)].[10] (eyal, I can't find the exact references – we need to add them!

Moreover, within the context of real estate, the availability of transaction (market) prices is often limited and thus the Shapley mechanism may generate prices when those are unobservable (correspondingly, an organization with several profit centers that are located on different stories of a building may use the Shapley mechanism to determine the rent cost allocation). Also, because of its fairness and efficiency properties, the Shapley cost allocation may provide a benchmark for the *fair* and *efficient* allocation. As previously mentioned, this allocation further exhibits properties that are consistent with relative status.

It should be noted that the Shapley solution has been applied to several practical problems in economics where allocation among players in coalitions is involved.

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[8] For an extensive survey of the Shapley value literature, see, for example, Roth (1988). Also, more on the interpretations of the Shapley axioms in a context similar to ours, see Dubey (1992).

[9] We propose here a somewhat different approach to the formulation of status than those existing in the literature. See the methodologies in, for example, Fershtman and Weiss (1993), Quint and Shubik (2001a, 2001b), and Luttmer (2004).

[10] Also, see Hart

Notably, Littlechild and Thompson (1977) show that the Shapley value approach to the allocation of the common costs of runway construction and landing fees highly resembles the actual charges employed at Birmingham airport during the investigated period (the years 1968-1969).[11]

In Section 2 we derive the Shapley value mechanism for allocating the land and construction cost and examine its properties both in closed-form and by simulation. In Section 3 we develop the Relative L&J Status function associated with the occupancy of different stories in the building and link its properties to the Shapley cost allocation solution. In section 4 we empirically test the major predictions derived under the theoretical framework. We summarize in Section 5.

2 The Shapley Solution of the Land and Construction Cost Allocation

Consider an N-story building. Let FC be the fixed cost associated with the construction of the entire structure. This cost consists of land, foundations, infrastructure, and all other elements that equally serve all the units in the building, independently of their specific vertical location in the building.[12] Further, denote the marginal cost that corresponds to the construction of story i by $mc(i)$, $i=1,\ldots,N$.[13]

It follows that for any N-story building, the total cost to be allocated among the different floors $TC\{N\}$, is

(1)

[11] More on the Shapley approach to the allocation of runway construction costs, see Littlechild and Owen (1973). Also, Loehman and Whinston (1971) present a pricing scheme for public utilities that, although not derived by the Shapley mechanism, bears some resemblance—as noted by the authors—to the Shapley solution. Finally, theoretical research in real estate economics that also examines aspects of construction includes Sullivan (1989) and, more generally, on the economics of construction see, for example, Vandell and Lane (1989) and Hysom and Crawford (1997).

[12] Other than the cost of placing the foundations and of setting up the common infrastructure, the fixed cost function, FC, might also include, for example, the cost of building various facilities for the common and equal service of all occupants. Note, however, that the fixed cost function may, in general, be discontinuous. E.g., while one stairway often suffices a 10-story building, additional floors might eventually necessitate the construction of another stairway. Similarly, the foundations laid for a 10-story building are likely to be less costly than those required for a 50-story structure. Therefore, in FC we only include those costs that involve the construction of elements that commonly and equally serve all stories and units in the structure. Other costs are to be included in the marginal cost component.

[13] Following the previous footnote, $mc(i)$ thus represents both the specific marginal cost associated with the construction of the individual story as well as the average cost that corresponds to the construction of the additional group of stories to which floor i belongs. If, for example, any additional group of 10 stories requires the construction of another stairway, then $mc(i)$ equals both the particular marginal cost of constructing the i-th floor as well as one-tenth of the total cost of constructing another stairway.

$$TC\{N\} = FC + \sum_{j=1}^{N} mc(j),$$

where we also assume that

(2)

$$TC\{\phi\} = 0,$$

(3)

$$TC\{S\} = \underset{i \in S}{Max}(TC\{i\}),$$

and

(4)

$$TC\{i\} = FC + \sum_{j=1}^{i} mc(j),$$

where S is any possible coalition of stories in the building and i, i=1,…,N, denotes a specific story in the building.[14]

Equation (1) implies that the total cost associated with the construction of a building simply equals the combination of fixed cost (cost of land, foundations, infrastructures, etc.) and marginal cost (incremental cost that corresponds to each additional story). Equation (2) guarantees that zero floors convey zero costs, and Equation (3) together with Equation (4) state that the total cost of constructing any subset of floors is equal to the total cost of constructing the highest floor within the subset.[15]

Following Shapley (1953), the Shapley value assigned to story j, φ_j, in an N-story building is defined by

(5)

$$\varphi_j = \sum_{S \subset N} \frac{(N-s)!(s-1)!}{N!}[TC\{S\} - TC\{S-\{j\}\}] \text{ for } j=1,\ldots N,$$

where s is the number of stories in subset S and φ_j is the Shapley value associated with story j.

The Shapley value that corresponds to story j, φ_j, can be interpreted as the expectation of the marginal contribution of the j-th story to the total construction cost,

[14] Formally, the assumption in Equation (1) is, of course, a specific case of Equation (4).

[15] Alternatively, S may be viewed as the current demand for the different stories, in which case Equation (3) represents the total construction cost, given that demand.

where the distribution of coalitions (any possible demanded sub-group of stories) arises in a particular way. To get a sense for φ_j, one can assume that the demand for floors may appear in any arbitrary order and that all N! orderings are equally likely. Then, φ_j is the expected value, across all possible orderings, of story j's marginal contribution to the total construction cost.

We now use the solution of Littlechild and Owen (1973) to re-write Equation (5). It should be noted, however, that our cost structure includes a fixed cost element, which is omitted from the setup of Littlechild and Owen (1973). Yet, since the fixed cost in our framework is used for factors that equally serve all occupants in the building, we get that for the cost function presented in Equations (1)-(4), the Shapley value in Equation (5) can be simplified into

(6)

$$\varphi_j = \sum_{k=1}^{j} \frac{TC\{k\} - TC\{k-1\}}{N-k+1} \text{ for } j=1,\ldots,N,$$

which can be further simplified into

(7)

$$\varphi_j = \varphi_{j-1} + \frac{mc(j)}{N-j+1} \text{ and } \varphi_1 = \frac{FC + mc(1)}{N}$$

Now, let i and j be two arbitrary stories in an N-story building, where $j>i$, (j=2,…,N, and i=1,…,N–1). We define the ratio between the cost allocated to the j-th story and that allocated to the i-th story according to the Shapley solution by $R(j,i,N)$ and the difference between the cost allocated to the j-th story and that allocated to the i-th story by $D(j,i,N)$. That is, $R(j,i,N) = \varphi_j / \varphi_i$ and $D(j,i,N) = \varphi_j - \varphi_i$.

Following Equations (1), (6), and (7), we argue

Proposition 1: *If $mc(i) \neq 0$ for all i=1,...,N, then both $R(j,i,N)$ and $D(j,i,N)$ increase with j and decrease with i, ceteris paribus.*

Proof: See appendix.

Proposition 1 implies that, according to the Shapley value approach to allocating the land and construction cost, the total cost of an N-story building is simply allocated to each story according to its vertical location in the building: the higher the floor in the building, the greater its allocated cost. The ratio and the difference functions provide a practical tool for comparing the floor price among different stories and buildings.

It further follows from Equation (7) that each floor carries the true pro rata cost associated with its construction: the cost associated with constructing the first floor is simply its fair share in FC and $mc(1)$. However, since the construction of the second floor relies on the construction of the first floor, the cost associated with the second floor is its fair share in FC, $mc(1)$, and $mc(2)$. More generally, the cost associated with the construction of the j-th floor is its fair share in FC, $mc(1)$, $mc(2)$, and up to $mc(j)$.

It thus turns out that the costs FC and $mc(1)$ are equally divided among all stories in the building, $mc(2)$ is divided among all stories from the second and beyond, and $mc(j)$ is divided among all stories from the j-th and beyond.

For example, suppose N=3. Then, according to the Shapley value in Equations (6) and (7), the cost allocated to the first floor, φ_1, is

$$\varphi_1 = \frac{FC + mc(1)}{3},$$

the cost allocated to the second floor, φ_2, is

$$\varphi_2 = \frac{FC + mc(1)}{3} + \frac{mc(2)}{2},$$

and the cost allocated to the third and last floor, φ_3, is

$$\varphi_3 = \frac{FC + mc(1)}{3} + \frac{mc(2)}{2} + mc(3),$$

where note that $\varphi_1 + \varphi_2 + \varphi_3 = TC\{3\}$.[16]

[16] As previously discussed, an alternative approach for attaining this cost allocation is to average story j's marginal contribution to the total construction cost across all possible orderings. For example, for the N=3 case:

Possible Ordering	Cost Allocated to Story:		
	1	2	3
1,2,3	$FC+mc(1)$	$mc(2)$	$mc(3)$
1,3,2	$FC+mc(1)$	0	$mc(2)+mc(3)$
2,1,3	0	$FC+mc(1)+mc(2)$	$mc(3)$
2,3,1	0	$FC+mc(1)+mc(2)$	$mc(3)$
3,1,2	0	0	$FC+mc(1)+mc(2)+mc(3)$
3,2,1	0	0	$FC+mc(1)+mc(2)+mc(3)$
Average	$[FC+mc(1)]/3$	$[FC+mc(1)]/3+mc(2)/2$	$[FC+mc(1)]/3+mc(2)/2+mc(3)$

In Figure 1 in the appendix, we demonstrate the costs allocated among the stories in a 50-story building. We compare the cost allocation for different shares of fixed cost, *FC*. Without loss of generality, we assume that the total cost of construction of the 50-story building (including land cost) is fixed at 100 million dollars. One can see that while the allocated cost curve always rises with the story level, the curve flattens as the share of the fixed cost increases.[17]

In the remaining of the analysis we assume that $mc(i)=mc(j)=mc\neq 0$ for all $i,j=1,\ldots,N$. We now claim

Proposition 2: *If* $\sum_{k=N-j+1}^{N}\frac{1}{k}>1-\frac{FC}{N\times mc}$ *(*$\sum_{k=N-j+1}^{N}\frac{1}{k}<1-\frac{FC}{N\times mc}$*), then* $R(j+1,j,N)$ *increases (decreases) with j, ceteris paribus.*

Proposition 3: $D(j+1,j,N)$ *monotonically increases with j, ceteris paribus.*

Proof: See appendix.

Interestingly, Proposition 2 entails that despite the fact that the Shapley value approach monotonically allocates greater land and construction cost to higher stories, the percentage change between the costs allocated to any two consecutive floors is not monotonic. In fact, in the general case, the ratio between the costs allocated to two successive stories first falls and then rises as one climbs along the floors of a given structure.

Moreover, the minimum of the ratio function between the costs allocated to any two successive stories is attained at the lowest j-th floor that sustains $\sum_{k=N-j+1}^{N}\frac{1}{k}>1-\frac{FC}{N\times mc}$.

In Figure 2 in the appendix, we demonstrate the additional cost allocated to any floor compared to that allocated to the preceding one (measured in percentage of the cost

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Also, note that the attained Shapley allocation is in the core of this cooperative game. That is, no coalition of stories can "gain" by dismissing this cost allocation and solely assume its cost of construction.

[17] A simple computation indicates that when the land cost captures 50% of the total cost, and further, when the cost of the foundations, infrastructure, etc. capture 50% of the remaining cost, then *FC* carries altogether 75% of the total land and construction cost of the building.

allocated to the preceding floor) for a 50-story building. The results are simulated and presented for a fixed cost share of 25%, 50%, and 75% of the total cost.

One can see that the percent change in the cost allocated to any floor relative to the previous one is contingent upon the share of the fixed cost. For example, while the function monotonically increases when the fixed cost share equals 50% and 75%, it first falls and then rises for a fixed cost share of 25%.[18]

We further argue

Proposition 4: *When N is large (i.e. for tall buildings), then R(j,i,N) converges to* $\frac{j+(FC/mc)}{i+(FC/mc)}$ *and D(j,i,N) converges to zero.*

Proposition 5: *The land and construction cost allocated to the j-th story converges to* $\frac{FC}{N}+\ln(\frac{N}{N-j})^{mc}$.

Proof: See appendix.

Proposition 5 thus provides a quick and simple method for computing the cost to be allocated to each story according to the Shapley value approach.

Note that the difference $H(N)–H(N–j)$ that appears in Equation (A19) increases with j at a relatively slow pace, implying that while the land and construction cost allocated to each story monotonically rises with the story's level, the growth is relatively slow.

We further examine the effect of the number of stories in the building on the derived Shapley solution. We claim

[18] Moreover, in a realistic case in which *FC* carries 75% of the total land and construction cost, the difference in the cost allocated to any two consecutive stories in the first 40 stories of the building is between 0.68% and 2.03%. These figures grow slowly to a 5.88% difference between the 47-th and 48-th stories, and finally jump to 15.39% difference between the 50-th (top story) and the 49-th stories.

Proposition 6: *If FC is a function of N, then, for non-increasing FC/N, the land and construction cost allocated to any j-th story drops with the number of stories in the building, ceteris paribus.*

Proof: See appendix.

According to common real estate economic literature, two stories exhibiting identical features (including the same height) and located in different building of identical quality should be allocated the same land and construction costs. Our approach, however, predicts otherwise—namely, that the allocated cost decreases with the number of stories in the building—and the intuition emerges from the L&J Relative Status function as will be further clarified in the next section.

We further argue

Proposition 7: *If FC is a function of N, then, for non-increasing FC/N, both R(j,i,N) and D(j,i,N) drop with the number of stories in the building, ceteris paribus.*

Proposition 8: *R(j,i,N) drops with and D(j,i,N) is independent of the level of the fixed cost within the total land and construction cost, ceteris paribus.*

Proof: See appendix.

Proposition 8 essentially states that as the level of *FC* within the total cost increases, the discrepancies in the costs allocated to different stories drops. This further implies that the average percent change in the cost allocated to all couplets of succeeding stories within a given building falls with the level of *FC*, *ceteris paribus*.

Figure 3 depicts the average percent change in the cost allocated to every two succeeding stories as a function of the number of stories in a given building. We replicate this simulation for varying shares of the fixed cost.

One can see that, independently of the share of the fixed cost, the average percent change in the cost allocated to all couplets of successive stories falls with the number of stories in the building. Furthermore, as we increase the share of the fixed cost, the average percent change in the cost allocated to all couplets of successive floors falls for any given building height.

Also,

Proposition 9: *For all levels of $FC \neq 0$, both $R(j,i,N)$ and $D(j,i,N)$ rise with the level of the marginal cost, ceteris paribus.*

Proof: See appendix.

Hence, from Propositions 8 and 9 it follows that $D(j,i,N)$ is independent of FC and rises with mc. This, in turn, implies that housing units that are located on different stories in more luxurious neighborhoods (where both FC and mc are relatively high) are likely to exhibit a greater absolute price difference.

Following the analysis of the Shapley approach to the land and construction cost allocation, we examine in the next section the relationship between the latter and the Relative L&J Status function.

3 The Relative L&J Status Function and the Shapley Solution

Consider an N-story building. Suppose that, except for the difference in their vertical location, all stories in the building are identical. However, due to its particular vertical location, each story produces (for its occupants) a different level of status. Following L&J, we define the *Relative L&J Status* associated with occupying the j-th story compared to occupying the i-th story in an N-Story building by $S(j,i,N)$.

We assume that the Relative L&J Status function, $S(j,i,N)$, is defined for all whole numbers, j, i, and N, where $N \geq j > i \geq 1$ and that its range are positive numbers. We then require that $S(\cdot)$ conforms to the following three axioms:[19]

Axiom 1: "*Path independence*." That is,

$$S(j+k,i,N)=S(j+k,j,N)+S(j,i,N) \text{ for all } k=1,2,\ldots.$$

Axiom 2: "*Marginal increase*." That is,

$$S(j+k,i+k,N)>S(j,i,N) \text{ for all } k=1,2,\ldots$$

[19] Consistent with L&J, the status is attained only from the attributes of the asset and not from its price, even though the relative price of any two stories is obviously contingent upon their relative status level.

Axiom 3: "*Additivity invariance.*" That is,

$$S(j+k,i+k,N+k)=S(j,i,N) \text{ for all } k=1,2,\ldots$$

Axiom 1 requires that the relative status function exhibits the additivity property. In other words, that relocating from a lower story to a higher one supplements the same level of L&J status independently whether the shift is performed directly from one floor to another or indirectly via an intermediate floor.

Axiom 2 requires that the Relative L&J Status function is marginally increasing. That is, that the additional status produced by any given shift from a lower to a higher story rises as the relocation commences at a higher story in the building.[20]

Finally, Axiom 3 requires that the Relative L&J Status is preserved for a fixed positive change in all elements of the function domain. That is, increasing stories j and i as well as the total number of stories in the building by a fixed sum preserves the level of Relative L&J Status.[21]

Following the above three Axioms, we claim

Corollary: *For all $k=1,2,\ldots$, Axioms 1-3 imply that $S(j+k,i,N)>S(j,i,N)$, $S(j,i+k,N)<S(j,i,N)$, and $S(j,i,N)>S(j,i,N+k)$.*

Proof: Axiom 1 combined with the fact that $S(\cdot)$ is positive implies that $S(j+k,i,N)>S(j,i,N)$. Also, note that Axiom 1 can be re-expressed as $S(j,i,N)=S(j,i+k,N)+S(i+k,i,N)$ for all $k=1,2,\ldots,j-i$, however, the latter together with the fact that $S(\cdot)$ is positive implies that $S(j,i,N)>S(j,i+k,N)$. Finally, Axiom 2 together with Axiom 3 implies that for all $k=1,2,\ldots$, $S(j+k,i+k,N+k)<S(j+k,i+k,N)$ and hence $S(j,i,N+k)<S(j,i,N)$. □

Note that the properties of $S(\cdot)$ stated in the Corollary are consistent with L&J: $S(j+k,i,N)>S(j,i,N)$ and $S(j,i+k,N)<S(j,i,N)$ for all $k=1,2,\ldots$ imply that the relatively higher is the j-th story compared to the i-th story, the greater (worse) the Relative L&J Status experienced by the occupant of the j-th (i-th) story. Furthermore, $S(j,i,N)>S(j,i,N+k)$

[20] This implies that relocating from one story to another is more meaningful (both status wise and, as we see later, cost wise) as it occurs closer to the top of the building. We will return to this point in Propositions 1, 6, and 10 as well as the empirical section.

implies that, while *j* always generates a greater status than *i*, its relative status drops, *ceteris paribus*, as more stories are added to the building. From the latter, however, it follows that the relative status function is sensitive to "irrelevant alternatives:" while the choice between any two given stories is independent of the total number of stories in the building, the attained Relative L&J Status is clearly affected.[22]

Interestingly, the difference in the costs allocated to any two stories according to the Shapley solution *D(j,i,N)* exhibits the properties of the Relative L&J Status function. Specifically, we claim

Proposition 10: *D(j,i,N) sustains Axioms* 1-3.

Proof: See appendix.

Proposition 10 thus asserts an important aspect of the Shapley approach to the cost allocation. Namely, that the difference between any two allocated costs conforms to the properties of the Relative Status function.

Interestingly, however, this further entails that the Shapley cost difference effectively provides an interpretation of the Relative L&J Status between two stories. For example, given Proposition 9, one can deduce that the Relative L&J Status between any two stories is higher when the cost of the building rises, that is, in more luxurious neighborhood.

4 Empirical Test

We propose an empirical model to test the properties of the Relative Lakoffian Status function and the Shapley solution discussed in the previous sections. Following volumes of literature on housing price we assume that house prices follow a log normal distribution (see, for example, Case and Shiller, 1989). We specify a hedonic housing price model such that

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[21] This implies that, in essence, two factors affect the relative status attained by any story: its distance from the lower story to which it is compared and its distance from the top (calculated by the number of stories).

[22] That is, the number of stories above and below a given floor directly affects the Relative L&J Status it generates.

(8)

$$\ln(P_k) = k\alpha_k + X\beta_k + Y_k\gamma_k + Z_k(t) + \varepsilon_k,$$

where P_k is the unit price of a condominium housing located on the *k*th floor of the building, *k* is the floor level, *X* is a vector of common building characteristics shared by condominium in all floor levels, such as number of stories in the building, *Y* is a vector of time invariant hedonic characteristics of the condominium, $Z(t)$ is a vector of time varying factors that may reflect the time trend in the housing market, and ε is *iid* normally distributed residuals.

Following Blinder (1973), we propose a structure estimate of our tests for Shapley solution for the allocation of the land and construction costs, such that

(9)

$$\ln\left(\frac{P_j}{P_i}\right) = (j\alpha_j - i\alpha_i) + X(\beta_j - \beta_i) + (Y_j\gamma_j - Y_i\gamma_i) + (Z_j(t) - Z_i(t)) + \omega,$$

where $j > i$, and $\omega = \varepsilon_j - \varepsilon_i$, which follows *iid* normal distribution.

Equation (37) can be re-arranged as

(10)

$$\ln\left(\frac{P_j}{P_i}\right) = (j-i)\alpha_i + j(\alpha_j - \alpha_i) + X(\beta_j - \beta_i) + (Y_j - Y_i)\gamma_i + Y_j(\gamma_j - \gamma_i)$$
$$+ (Z_j(t) - Z_i(t)) + \omega.$$

Let $\alpha_i = \theta_1$, $(\alpha_j - \alpha_i) = \theta_2$, $(\beta_j - \beta_i) = \theta_3$, $\gamma_i = \theta_4$, $(\gamma_j - \gamma_i) = \theta_5$, $X = N$ and $(Z_j(t) - Z_i(t)) = I(t)$. Equation (38) can then be expressed as

(11)

$$\ln\left(\frac{P_j}{P_i}\right) = (j-i)\theta_1 + j\theta_2 + N\theta_3 + (Y_j - Y_i)\theta_4 + Y_j\theta_5 + I(t) + \omega,$$

where $(j-i)$ is the floor level difference of the two condominium units, j is the floor level of the condominium, N is the number of stories in the building, $(Y_j - Y_i)$ captures the difference in hedonic characteristics between the condominium units located on the jth floor and the units located on the ith floor, and $I(t)$ may be the house price index or dummy variables that capture the time trend in the housing market. The key coefficients for our empirical tests are θ_1, θ_2, and θ_3. Following the Relative Lakoffian Status properties and Shapley cost allocation discussed in previous sections, we expect θ_1 and θ_2 to be positive, and θ_3 to be negative.

We collect a sample of newly developed condominium sales data from two major cities in China. The sample contains 924 condo units from 16 different buildings sold during the period of 2004 to 2005. These buildings vary from 6-story to 33-story. The sales transaction data contains information including sales price (per square meter), date of sale, building ID, unit number, floor level where the condo unit is located, number of stories in the building, size of the condo unit, as well as other hedonic characteristics of the unit, such as number of bedrooms, number of living rooms, etc.

We construct a transaction-pairs data by comparing any two transactions of condo units located on two different floor levels within each building. The final sample contains 38,926 pairs of sales transactions. Each paired observation contains sales information of the two comparing condo units, including the log of the per-square-meter price ratio between the condo on the jth floor and condo on the ith floor, the floor levels of the two condo units, number of stories in the building where the two condo units located, date of each transaction, different hedonic characteristics between the two condo units, such as size difference, number of bedrooms, number of living rooms, etc.

Table 1 presents two sets of simplified models to test Shapley value allocation property. Model 1 and model 2 test the proposition 2: following Shapley value allocation rule, the price ratio between floors j and j + 1 first decreases with j for lower stories then increases with j for higher stories, *ceteris paribus*. We confine our sample to those transaction pairs where $j = i + 1$. The restricted sample contains 2,867 pairs of condo sales from neighboring floors. Empirical results from model 1 support proposition 2 that price ratio between condo located on floors j and j + 1 is a concave function of j, i.e., it first decrease then increases with j, at 1% significant level, *ceteris paribus*. Model 2 extends model 1 by controlling for time dummies of sales. Same test result holds. Model 3 and model 4 regress the price difference between condo located on floors j and j + 1. The results from model 3 and 4 support proposition 3 that the price difference between floors j and j + 1 increases monotonically with j for higher stories, *ceteris paribus*. All these tests are significant at 1% level.

Table 2 presents a set of extended empirical tests such that price ratios between floors i and j increases with $(j - i)$, $j > i$, *ceteris paribus*. Models 5-8 in table 2 are estimated based on the entire 38,926 condo sales pairs. Model 3 follows equation (39). We assume $\gamma_j = \gamma_i$, hence $\theta_5 = 0$. In other words, the hedonic value of a two-bedroom unit located on the ith floor is identical to the hedonic value of a two-bedroom unit on the jth floor, *ceteris paribus*.[23] Model 5 further assumes that there is no significant time trend during our sampling period of 2004 to 2005 in Chinese housing market. As expected, price ratio increases with the floor level difference, $(j - i)$. The empirical results from model 5 support proposition 1 that price ratio increases with j, *ceteris paribus*. Results from model 5 further support proposition 7 that price ratio decreases with N, number of stories in the building, *ceteris paribus*. There are no surprises for the rest of the hedonic control variables. Per-square-meter price ratio increases with the condo size, but decreases with number of bedrooms and living rooms. All coefficients are significant at 1% level.

[23] Homebuyers in China prefer doing interior design and decoration according to individual tastes, needs and budget constraint; therefore most of the condominium units sold in China are delivered as a concrete shell without any interior installation. It is reasonable to assume that a two-bedroom unit on different floor levels within each building carries identical hedonic value. However, this assumption is not critical to our test results.

Model 6 extends model 5 by exploring the interactions of the floor level difference with floor level, j, and number of stories in the building, N. The empirical results reconfirm the properties stated in the proposition 1 and proposition 7, that the price ratio increase with j, and drops with N, the number of stories in the building, *ceteris paribus*.

Model 7 extends model 6 by adding a control for the units sold in 2005. While the key results on $(j-i)$, j and N remain the same as reported in model 6, the results also suggest a statistically significant positive impact on the price ratio for the units sold in 2005.

Model 8 extends model 7 by exploring the interaction of the floor level difference with floor level i. The empirical results support the Shapley allocation properties stated in proposition 1 and proposition 7 that price ratio increases with j, and decreases with i and N, *ceteris paribus*. All the estimates reported in table 2 are statistically significant at 1% level.

5 Summary

We apply the Shapley value approach for allocating the land and construction cost among the stories of a building. We also develop a status function of which properties align with Lakoff and Johnson (1980) imperative contribution to the perception of the terms "up" and "down."

According to the proposed cost allocation mechanism, each story's share in the total cost rises with its level in the building in a particular manner. We show, among other things: the conditions under which the ratio and the difference between the costs allocated to any two successive stories either rise or fall with an ascent to higher stories; that the land and construction cost allocated to any given story drops with the number of stories in the building (when total costs are increased proportionally); that the ratio and the difference between the cost allocated to any two stories drop with the number of stories in the building; and that that the ratio (difference) between the cost allocated to any two stories drops with (is independent of) the share of the fixed cost.

Interestingly, we further show that the difference between the cost allocated to any two distinct stories in the building coincides with a set of reasonable properties required from the Relative L&J Status function. The latter thus establishes a new property to the Shapley solution.

Our empirical tests support the Shapley cost allocation properties according to which the price ratio monotonically increases with j and decreases with i and N, *ceteris paribus*.

Our results incorporate several economic implications. From a normative aspect, we provide a benchmark for the fair allocation of the land and construction cost as a function of the various construction variables (fixed cost, marginal cost, and the number of stories in the building). The allocation may then generate prices when those are unobservable such as in the case of allocating costs in an organization (in both the public and private sectors) that shares diverse profit centers, however, where the profit centers occupy different stories within a common building. Finally, it turns out that the cost allocation may provide values for the relative status associated with occupying different stories in the building as it is shown to be consistent relative status properties. The latter may be applied, for example, by Courts in cases of tenant disputes regarding expansions and redevelopments.

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Appendix

Figure 1: The costs allocated to each story in a 50-story building. The results are shown for a fixed cost share of 25%, 50%, and 75% of the total land and construction cost and constant marginal costs.

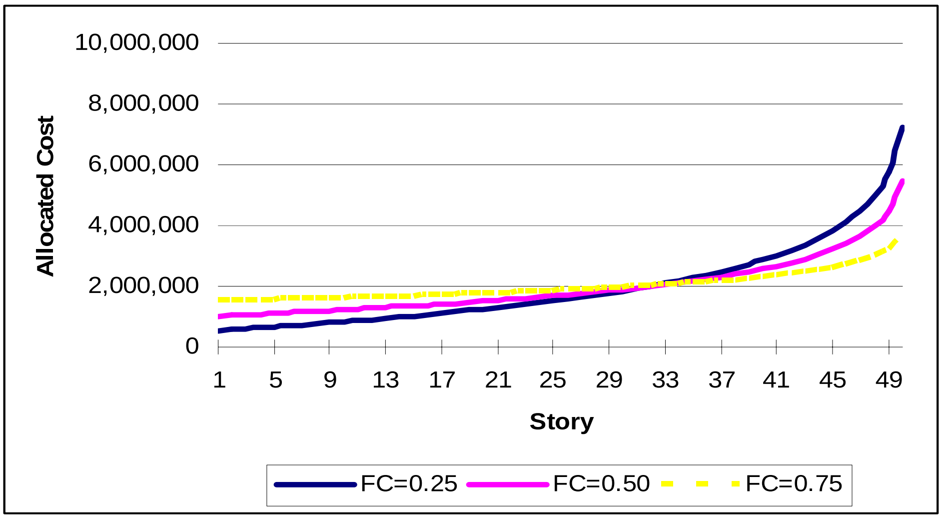

Figure 2: The additional cost allocated to any floor compared to the cost allocated to the preceding floor (measured in percentage difference from the cost allocated to the preceding floor) for a 50-story building. The results are shown for a fixed cost share of 25%, 50%, and 75% of the total land and construction cost and constant marginal costs.

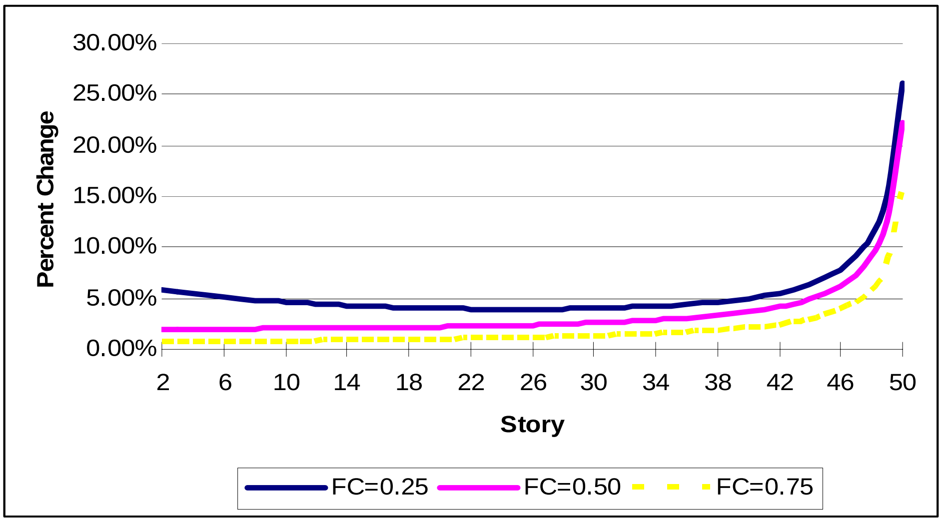

Figure 3: Average percent change in the cost allocated to any two succeeding stories as a function of the number of stories in the building. The results are shown for a fixed cost share of 25%, 50%, and 75% of the total land and construction cost and constant marginal costs.

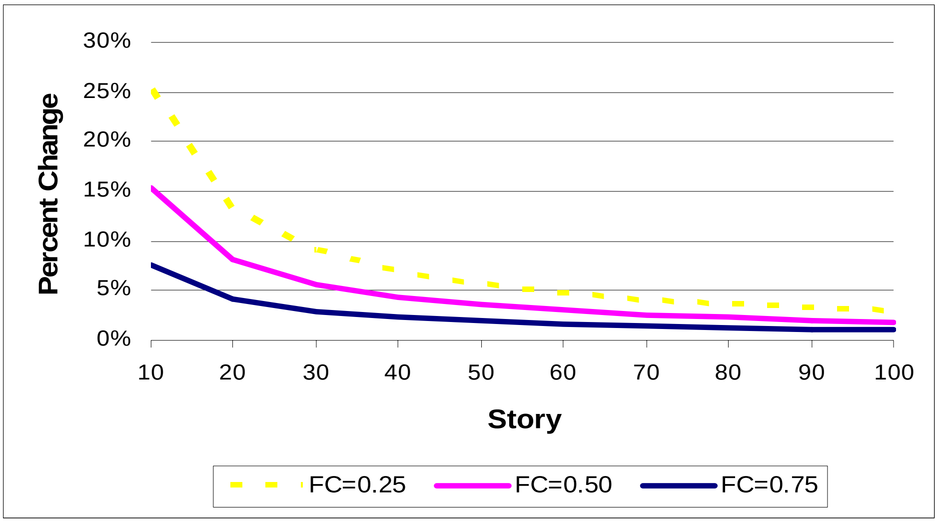

Table 1: The empirical tests of the Shapley Solution. Dependent variables of the regressions in models 1 and 2 (models 3 and 4) are log per-square-meter price ratio (per-square-meter price difference) between the unit at floor level *j* and the unit at floor level *i*. We confine the sample to those transaction pairs where $j = i + 1$. The empirical results support the Shapley cost allocation properties. Model 1 and model 2 indicate that the price ratio decreases for lower stories and increases for higher stories, while model 3 and model 4 suggest that the price difference monotonically increases with *j*, *ceteris paribus*. Standard errors are in parentheses. ** indicates significant at 1% level.

	Model 1	Model 2	Model 3	Model 4
Floor level (*j*)	-0.00307** (0.00063)	-0.00237** (0.00059)	10.36** (2.89)	12.30** (2.72)
Floor level square (j^2)	0.00014** (0.00002)	0.00011** (0.00002)		
Log unit size	0.10094** (0.01003)	0.09198** (0.00938)	2,209.36** (179.35)	2,004.75** (170.22)
Unit sold in 2004 (dummy variable)		0.15263** (0.00860)		2,233.76** (156.12)
Unit sold in 2005 (dummy variable)		0.22852** (0.01039)		3,663.31** (188.46)
No. of Observations	2,867	2,867	2,867	2,867
R^2	0.0491	0.1867	0.0574	0.1696

Table 2: The empirical tests of the Shapley Solution. Dependent variables of the regressions are log per-square-meter price ratio between the unit at floor level *j* and the unit at floor level *i*. The results support the Shapley cost allocation properties according to which price ratio between floor level *j* and *i* increases with *j* and decreases with *i*, and the ratio drops with *N*, the number of stories in the building. Standard errors are in parentheses. ** indicates significant at 1% level.

	Model 5	Model 6	Model 7	Model 8
Floor level difference (*j* - *i*)	0.00241** (0.00013)			
Floor level (*j*)	0.00167** (0.00013)			
Number of stories in the Building (*N*)	-0.00070** (0.00008)			
Floor level difference × *i*				-0.00014** (0.00002)
Floor level difference × *j*		0.00022** (0.00001)	0.00016** (0.00001)	0.00022** (0.00002)
Floor level difference × *N*		-0.00007** (0.00001)	-0.00004** (0.00001)	-0.00008** (0.00001)
Log unit size	0.47573** (0.00797)	0.47321** (0.00796)	0.40707** (0.00788)	0.40328** (0.00789)
Two bedroom unit (dummy variable)	-0.29924** (0.00612)	-0.30053** (0.00611)	-0.29271** (0.00596)	-0.29214** (0.00596)
Three bedroom unit (dummy variable)	-0.40996** (0.00820)	-0.41001** (0.00818)	-0.37729** (0.00801)	-0.37486** (0.00801)
Four bedroom unit (dummy variable)	-0.54392** (0.01117)	-0.54341** (0.01115)	-0.50116** (0.01090)	-0.49939** (0.01090)
Three living-room Unit (dummy variable)	-0.11369** (0.00733)	-0.11304** (0.00732)	-0.09469** (0.00714)	-0.09255** (0.00714)
Unit sold in 2005 (dummy variable)			0.07848** (0.00180)	0.08120** (0.00183)
No. of Observations	38,926	38,926	38,926	38,926
R^2	0.1944	0.1966	0.1836	0.1851

Proof of Proposition 1: Following Equations (1), (6), and (7), the Shapley value associated with any floor j, j=1,…,N, is

(A1)

$$\varphi_j = \frac{FC}{N} + \frac{mc(1)}{N} + \frac{mc(2)}{N-1} + \cdots + \frac{mc(j-1)}{N-j+2} + \frac{mc(j)}{N-j+1},$$

which may be re-expressed as

(A2)

$$\varphi_j = \frac{FC}{N} + \sum_{k=N-j+1}^{N} \frac{mc(N-k+1)}{k}.$$

Hence, the ratio between the cost allocated to the j-th story and that allocated to the i-th story is

(A3)

$$R(j,i,N) = \frac{\dfrac{FC}{N} + \sum_{k=N-j+1}^{N} \dfrac{mc(N-k+1)}{k}}{\dfrac{FC}{N} + \sum_{k=N-i+1}^{N} \dfrac{mc(N-k+1)}{k}}$$

and the difference between the cost allocated to the j-th story and that allocated to the i-th story is

(A4)

$$D(j,i,N) = \sum_{k=N-j+1}^{N} \frac{mc(N-k+1)}{k} - \sum_{k=N-i+1}^{N} \frac{mc(N-k+1)}{k}.$$

From Equations (A3) and (A4), however, it is straightforward to see that if $mc(i) \neq 0$ for all i=1,…,N, then both $R(j,i,N)$ and $D(j,i,N)$ increase (decrease) with j (i), *ceteris paribus*. □

Proof of Propositions 2 and 3: Note that when $mc(i)=mc(j)=mc$ for all i,j=1,…,N, then, it follows from Equation (A3) that $R(j,i,N)$ is

(A5)

$$R(j,i,N) = \frac{\dfrac{FC}{N \times mc} + \sum_{k=N-j+1}^{N} \dfrac{1}{k}}{\dfrac{FC}{N \times mc} + \sum_{k=N-i+1}^{N} \dfrac{1}{k}}.$$

Following Equation (A5), one can see that the ratio between the cost allocated to the (j+1)-th floor and that allocated to the j-th floor increases with j, *ceteris paribus*, if and only if

(A6)

$$\frac{\frac{FC}{N \times mc} + \sum_{k=N-j+1}^{N} \frac{1}{k}}{\frac{FC}{N \times mc} + \sum_{k=N-j+2}^{N} \frac{1}{k}} < \frac{\frac{FC}{N \times mc} + \sum_{k=N-j}^{N} \frac{1}{k}}{\frac{FC}{N \times mc} + \sum_{k=N-j+1}^{N} \frac{1}{k}}.$$

However, the latter inequality yields

(A7)

$$1 + \frac{\frac{1}{N-j+1}}{\frac{FC}{mc \times N} + \sum_{k=N-j+2}^{N} \frac{1}{k}} < 1 + \frac{\frac{1}{N-j}}{\frac{FC}{mc \times N} + \sum_{k=N-j+1}^{N} \frac{1}{k}},$$

which may, in turn, be re-expressed as

(A8)

$$\frac{FC}{N \times mc}\left(\frac{1}{N-j+1} - \frac{1}{N-j}\right) < \frac{1}{N-j}\sum_{k=N-j+2}^{N} \frac{1}{k} - \frac{1}{N-j+1}\sum_{k=N-j+1}^{N} \frac{1}{k}.$$

Inequality (A8) may be further simplified into

(A9)

$$\frac{FC}{N \times mc}\left[\frac{-1}{(N-j+1)(N-j)}\right] < \sum_{k=N-j+2}^{N} \frac{1}{k}\left(\frac{1}{N-j} - \frac{1}{N-j+1}\right) - \frac{1}{(N-j+1)^2}.$$

Multiplying both sides of (A9) by $(N-j+1)(N-j)$ produces

(A10)

$$\frac{-FC}{N \times mc} < \sum_{k=N-j+2}^{N} \frac{1}{k} - \frac{N-j}{(N-j+1)},$$

implying that

(A11)

$$\frac{N-j}{N-j+1} - \frac{FC}{N \times mc} < \sum_{k=N-j+1}^{N} \frac{1}{k} - \frac{1}{N-j+1},$$

and therefore that

(A12)

$$1-\frac{FC}{N\times mc}<\sum_{k=N-j+1}^{N}\frac{1}{k}.$$

Hence, if $\sum_{k=N-j+1}^{N}\frac{1}{k}>1-\frac{FC}{N\times mc}$ $(\sum_{k=N-j+1}^{N}\frac{1}{k}<1-\frac{FC}{N\times mc})$, then the ratio between the cost allocated to the (j+1)-th floor and that allocated to the j-th floor increases (decreases) with j, *ceteris paribus*.

Also when $mc(i)=mc(j)=mc$ for all $i,j=1,\ldots,N$, then, from Equation (A4) it follows that $D(j,i,N)$ is

(A13)

$$D(j,i,N)=mc(\sum_{k=N-j+1}^{N}\frac{1}{k}-\sum_{k=N-i+1}^{N}\frac{1}{k}).$$

Substituting j+1 and j for j and i, respectively, produces after reduction

(A14)

$$D(j,i,N)=\frac{mc}{N-j},$$

which monotonically increases with j. □

Proof of Proposition 4: It follows from Equation (A5) that

(A15)

$$\frac{\frac{FC}{N\times mc}+\frac{j}{N-j+1}}{\frac{FC}{N\times mc}+\frac{i}{N}}\geq R(i,j,N)\geq\frac{\frac{FC}{N\times mc}+\frac{j}{N}}{\frac{FC}{N\times mc}+\frac{i}{N-i+1}},$$

where the left-hand side (right-hand side) of the first (second) inequality in (A15) is the maximal (minimal) value of the right-hand side of Equation (A5).

However, (A15) implies that

(A16)

$$\frac{\frac{(FC/mc)+j}{N-j+1}}{\frac{(FC/mc)+i}{N}}\geq R(i,j,N)\geq\frac{\frac{(FC/mc)+j}{N}}{\frac{(FC/mc)+i}{N-i+1}}.$$

Note that for a large N, the terms on both the left-hand side and the right-hand side of (A16) converge to $\frac{j+(FC/mc)}{i+(FC/mc)}$.

Also, from (A13) we have

(A17)

$$D(j,i,N)=mc\sum_{k=N-j+1}^{N-i}\frac{1}{k},$$

where the right-hand side of (A17) converges to zero with N. □

Proof of Proposition 5: Note that when $mc(i)=mc(j)=mc$ for all $i,j=1,\ldots,N$, then Equation (A2), may be re-expressed as

(A18)

$$\varphi_j=\frac{FC}{N}+mc\sum_{k=N-j+1}^{N}\frac{1}{k},$$

which may, in turn, be re-expressed as

(A19)

$$\varphi_j=\frac{FC}{N}+mc[H(N)-H(N-j)],$$

where $H(l)$ is the sum of the harmonic series $\sum_{k=1}^{l}\frac{1}{k}$.

It is known, however, that $H(l)\cong\ln(l)$. And thus Equation (A19) yields

(A20)

$$\varphi_j=\frac{FC}{N}+mc[\ln(N)-\ln(N-j)],$$

which further implies that

(A21)

$$\varphi_j=\frac{FC}{N}+\ln(\frac{N}{N-j})^{mc}.\ \square$$

Proof of Proposition 6: Focusing on the expression that appears on the right-hand side of Equation (A19), note that while $\frac{FC}{N}$ is assumed to be non-increasing with N, the difference $H(N)-H(N-j)$ drops with the value of N (where, once again, $H(l)$ is the sum of the harmonic series $\sum_{k=1}^{l}\frac{1}{k}$). □

Proof of Proposition 7: It follows from Equation (A5) that $R(j,i,N)$ ($j>i$; $i=1,\ldots,N-1$; $j=2,\ldots,N$, $j>i$), drops with the number of stories, N, if and only if

(A22)

$$\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}} > \frac{\frac{FC}{N\times mc}+\sum_{k=N-j+2}^{N+1}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+2}^{N+1}\frac{1}{k}}.$$

However, note that the right-hand side of Inequality (A22) can be developed into

(A23)

$$\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+2}^{N+1}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+2}^{N+1}\frac{1}{k}} = \frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}+(\frac{1}{N+1}-\frac{1}{N-j+1})}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}+(\frac{1}{N+1}-\frac{1}{N-i+1})}.$$

Substituting the right-hand side of Equation (A23) with the right-hand side of Inequality (A22) produces after reduction

(A24)

$$\frac{j(N-i+1)}{i(N-j+1)} > \frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}.$$

Given that $j>i$, we can re-write the right-hand side of (A24) to generate

(A25)

$$\frac{j(N-i+1)}{i(N-j+1)} > \frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N-i}\frac{1}{k}+\sum_{k=N-i+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}},$$

and thus

(A26)

$$\frac{j(N-i+1)}{i(N-j+1)}-1>\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}},$$

which can then be developed into

(A27)

$$\frac{(N+1)(j-i)}{i(N-j+1)}>\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}.$$

Focusing on the right-hand side of Equation (A27), one can see that

(A28)

$$\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}\leq\frac{\frac{(j-i)}{(N-j+1)}}{\frac{FC}{N\times mc}+\frac{i}{N}},$$

where the right-hand side of (A28) is the maximal value of the right-hand side of (A27).

Finally, note that the right-hand side of (A28) is smaller than the left-hand side of (A27), which thus implies that Inequality (A22) holds for all N and $j>i$, and hence that the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor ($j>i$; $i=1,\ldots,N-1$; $j=2,\ldots,N$, $j>i$) drops with the number of stories (for FC/N non-increasing with N).

Also, from (A17) it is immediate that $D(j,i,N)$ drops with N (for FC/N non-increasing with N). □

Proof of Proposition 8: It follows from Equation (A5) that the ratio $R(j,i,N)>1$ for all $i=1,\ldots,N-1$, $j=2,\ldots,N$, and $j>i$. Hence, as the share of FC increases, we get that $\frac{FC}{N\times mc}$ rises, and thus that $R(j,i,N)$ falls. Also, from (A13) it is immediate that $D(j,i,N)$ is independent of FC. □

Proof of Proposition 9: Following Equations (A5) and (A13) the result for $R(j,i,N)$ and $D(j,i,N)$, respectively, is immediate. □

Proof of Proposition 10: Following Equation (A17), Axiom 1 is maintained by definition because for all m=1,2,… we have $\varphi_{j+m} - \varphi_i = (\varphi_{j+m} - \varphi_j) + (\varphi_j - \varphi_i)$. Also, Axiom 2 is maintained because $\varphi_{j+m} - \varphi_{i+m} = \varphi_j - \varphi_i + mc(\sum_{k=N-j+m}^{N-j} \frac{1}{k} - \sum_{k=N-i+m}^{N-i} \frac{1}{k})$, where the term in the parenthesis $(\sum_{k=N-j+m}^{N-j} \frac{1}{k} - \sum_{k=N-i+m}^{N-i} \frac{1}{k}) > 0$. Finally, given Equation (A2), Axiom 3 is maintained:

$$\varphi_{j+m,N+m} - \varphi_{i+m,N+m} = mc(\sum_{k=N+m-j-m+1}^{N+m} \frac{1}{k} - \sum_{k=N+m-i-m+1}^{N+m} \frac{1}{k}) = mc(\sum_{k=N-j+1}^{N-i} \frac{1}{k}) = \varphi_j - \varphi_i\,,$$

where $\varphi_{j+m,N+m}$ is the cost allocated to the $j+m$ story in an (N+m)-story building. □